servations that would be of value for the furtherance of our knowledge as to the clouds of ice crystals in front of the whirlwind or as to the local similar and simultaneous formation of ice in the atmosphere.

Even the individuality of the observer has an influence upon the minute details of observations that is of an importance not to be underestimated, and, therefore, every observer should seek to understand the errors to which he is personally liable, and the times at which he is in the best personal condition. This has a special bearing on the perception and distinction between colors. To one person great differences of intensity will alone affect his consciousness, whereas another has the power of distinguishing the finest shades; therefore, a note upon the individual colors that are observed and their succession from the inside to the outside of the halo should not be neglected. Not less important is the difference in the observation of the blue tints on the outer side of the circle, since one observer can observe violet only in a narrow zone, whereas another can follow it throughout a wider band.

In conclusion the subscriber invites all who have sufficient leisure and love of nature to join in the minutest observations of halo phenomena, and to kindly send him the results, since the greatest usefulness is to be expected from the discussion of all the material from a single point of view, and since, moreover, individual observations are easily scattered and lost.

THE EQUATIONS OF HYDRODYNAMICS IN A FORM SUITABLE FOR APPLICATION TO PROBLEMS CONNECTED WITH THE MOVEMENTS OF THE EARTH'S ATMOSPHERE.

By JOSEPH COTTIER, Columbia University, N.Y. (dated June 29, 1897).

Introductory note.—The equations of fluid motion as usually given in terms of rectangular coordinates are unsuitable for use in problems connected with the motions of fluids on the earth's surface, owing to the curved surface of the latter. Hence, either the Cartesian equations must be transformed into the equations for polar coordinates, by purely mathematical considerations, or an independent deduction of the equations in terms of polar coordinates must be made. The former course was adopted by Mr. Basset[1] in obtaining the equations of an incompressible fluid in terms of polar coordinates; the latter process is that adopted in this paper, longer and more laborious perhaps, but having the advantage of giving a clearer conception of the terms entering the final equations and of the intermediate steps.

The plan of this article follows closely that of a paper read by the writer before the American Mathematical Society,[2] adapted to the simpler polar coordinates. Since it is hoped this latter paper will soon be published, many of the more complex reductions therein have been dismissed with a mere reference, as being unnecessary to a comprehension of the equations and methods here employed.

General considerations.—The earth is a slightly ellipsoidal body, whose mean radius is about 3,959 miles, or 6,371 kilometers; the polar diameter is, roughly, only one three-hundredth part shorter than the mean equatorial diameter, so that all necessary accuracy will be obtained by considering the surface of the earth as a sphere of the above radius.

The coordinate system suitable for rotating spherical surfaces, or for surfaces departing but little from the spherical form, is, of course, the well-known "polar coordinate" system, where the coordinates determining the position of a point are its north polar distance (θ), its rotation angle measured eastward (λ), and its distance from the center of the earth (r). The measurement of geographic longitude positive westward is determined by the fact of the eastward rotation of the earth in space.

[1] Treatise on Hydrodynamics. Vol. II. §470.

[2] "On the Expression of the General Equations of Hydrodynamics in Terms of Curvilinear Coordinates," read before the American Mathematical Society, March 27, 1897, and about to be published in The Mathematical Review.

The equations of motion which it is desired to obtain are three equations, which for three separate orthogonal directions express the equality of the "expressed force" of a fluid particle to the sum of the "impressed forces" and of the resultant surface traction on the particle in the same direction.

The "expressed force" of a particle is the force necessary to produce the existent acceleration in the given direction, and is equal to the product of the mass of the particle and of that acceleration.

The "impressed forces" are those forces which are applied individually to each particle of the fluid, such as gravitational forces, electrical forces, etc. The only impressed force which need be taken into account in connection with atmospheric disturbances is the gravitational attraction of the earth, whose value per unit mass in any direction may be expressed as the derivative in that direction of the Newtonian potential, F. It may be considered that the derivatives of F with respect to θ and λ, that is to say, parallel to the surface of the earth, are each zero.

The surface tractions are due in part to the "fluid pressure," constant in all directions at a given point and independent of the viscosity of the fluid, and in part to the internal friction of the fluid determined by the relative motions of its particles.

In all material fluids the internal friction or viscosity can not change the mean normal pressure, or the fluid pressure at any point, but it does cause a tangential resistance, or "shearing stress," between layers of particles which are in relative motion. It is generally assumed, and the assumption is borne out by experience, that the shearing stress at any point and parallel to any plane, is proportional to the rate at which the velocity parallel to the plane is varying in a direction normal to that plane, that is to say, viscosity is proportional to that component of the relative motion of the particles that is parallel to that plane, or to the rate of "shearing stress" upon it. A finite slip between the fluid and the surfaces with which it may be in contact, or between contiguous layers of the fluid itself, or between the layers at the limiting surface that is technically known as a "surface of discontinuity," would thus mean an infinite shearing stress parallel to the surface of discontinuity, and is hence precluded.

The coefficient of friction expresses the ratio between the shearing stress and the shearing strain; it may be defined conveniently as the difference of the tangential tractions per unit area on two very large and parallel planes at the unit distance apart, moving in the same direction with the unit difference of velocity. The space between these parallel planes being filled with the fluid, the velocity gradient in a direction normal to the slower moving plane will be unity, and therefore, according to definition, the intensity of the tangential stress on that plane due to viscosity, will be numerically equal to the coefficient of viscosity, which by common consent is denoted by the Greek letter μ. The coefficient of viscosity is, approximately, proportional directly to the absolute temperature of the air, and is independent of the pressure.[1] As nearly as present experimental data permit of its estimation, the value of the coefficient of viscosity at a temperature of 62° F., in terms of British units,[2] may be considered as being:

[1] Maxwell, Phil. Trans., 1863.

[2] The British units referred to are the foot and mean solar second as units of length and time, and the mass of a pound of matter as the unit of mass. The unit of force, or "poundal," is the force which, acting on a unit mass, produces unit acceleration.

$$\mu = 0.0000125 \text{ lbs. / ft. sec.},$$

or, in the C. G. S. system,

$$\mu = 0.000185 \text{ gr. / cm. sec.}$$

From the nature of μ it will be seen that its dimensions are $M^{+1} L^{-1} T^{-1}$.

The equations of motion are applied to, and the coefficient of friction is defined for, either so-called steady motion or motion varying according to some definite law; whereas the atmosphere is in a constant state of commotion and of irregular movement. M. Boussinesq[1] has shown that in the case of turbulent motion the general equations of the mean motion take the same forms as the equations for steady or regularly varying motion, provided the coefficient of friction be considered variable and dependent upon the local turbulence of the movement. If the gusty character of the atmosphere be quite universally and uniformly distributed, it may be presumed that a sufficient allowance for it may be made by slightly increasing the value of μ.

Kinematical considerations.—It is usual among meteorologists to define the motion of an atmospheric particle by the three components of the velocity, counted positively southward, eastward, and zenithward, denoted by u, v, and w, respectively. The spacial displacements measured from a given point in these directions are similarly denoted by x, y, and z. These symbols will be adhered to as far as possible in this paper. These velocity components are derived quantities, being connected with the space-rates of variation of the coordinates by the equations:

$$u = r\, d\theta/dt \ldots\ldots\ldots\ldots \text{(southward)}.$$
$$v = r \sin\theta\, d\lambda/dt \ldots\ldots\ldots \text{(eastward)}.$$
$$w = dr/dt \ldots\ldots\ldots\ldots\ldots \text{(zenithward)}.$$

It will be noted that the terms "southward," "eastward," "zenithward" only have a definite meaning when the position of the particle on the terrestrial sphere is specified. They mean different absolute directions for any particles not on the same radius.

All the kinematical quantities at present sought may be made to literally drop into our hands from the following simple fiction: Let two auxiliary systems of Cartesian (rectangular and rectilinear) axes be chosen; let the one system, XYZ (identical with that previously mentioned if the rotation of the earth be neglected), be fixed in space and coincident respectively with the southward, eastward, and zenithward directions at the position P_0 occupied by the centroid of a particle at a given time $t = t_0$; let the second system, $X_1 Y_1 Z_1$, also have its origin fixed in space at P_0. If then to this second system, which may be supposed to have directional freedom, there be imparted constant angular velocities

$$\omega_1 = -v/r; \quad \omega_2 = +u/r; \quad \omega_3 = +v/r \cot\theta,$$

counted positive in the cyclic directions (left-handed), these moving axes will, for a small interval of time immediately preceding or following the given instant, remain parallel to the southward, eastward, and zenithward directions corresponding to the position P occupied by the centroid of the particle at the time t. It is assumed that at the given instant P is moving away from the position P_0, at a velocity whose components resolved along X, Y, and Z are u, v, and w, respectively. (See Fig. 1.)

It is convenient to use the Newtonian notation, $\dot{x}, \dot{y}, \dot{z}$, for the components of the absolute velocity of a point, and $\ddot{x}, \ddot{y}, \ddot{z}$, for the components of its absolute acceleration, referred to the fixed Cartesian system, XYZ, with a similar notation for the relative velocity and acceleration with respect to the moving system, $X_1 Y_1 Z_1$. Since these two systems of axes coincide at the initial instant, the ordinary laws of relative motion show that

$$\dot{x} = \dot{x}_1 - y_1 \omega_3 + z_1 \omega_2$$
$$\dot{y} = \dot{y}_1 - z_1 \omega_1 + x_1 \omega_3$$
$$\dot{z} = \dot{z}_1 - x_1 \omega_2 + y_1 \omega_1$$

and remembering that the angular velocities of the secondary system are constant, upon differentiation with respect to the time,

$$\ddot{x} = \ddot{x}_1 - \dot{y}_1 \omega_3 + \dot{z}_1 \omega_2$$
$$\ddot{y} = \ddot{y}_1 - \dot{z}_1 \omega_1 + \dot{x}_1 \omega_3$$
$$\ddot{z} = \ddot{z}_1 - \dot{x}_1 \omega_2 + \dot{y}_1 \omega_1$$

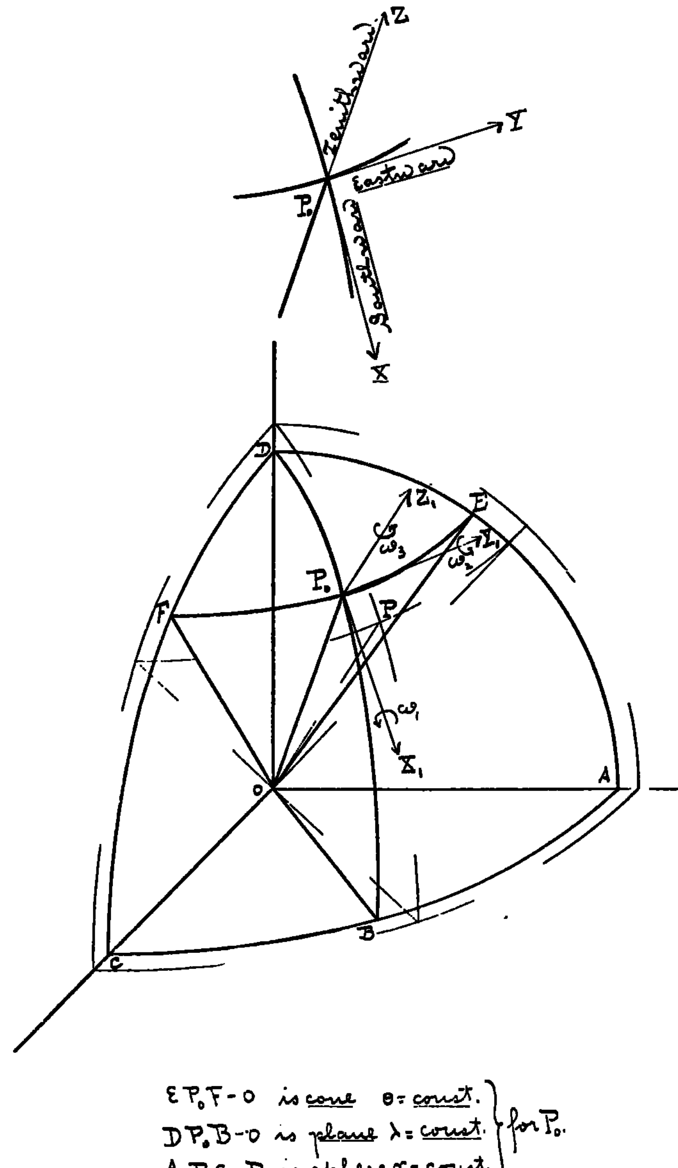


FIG. 1.

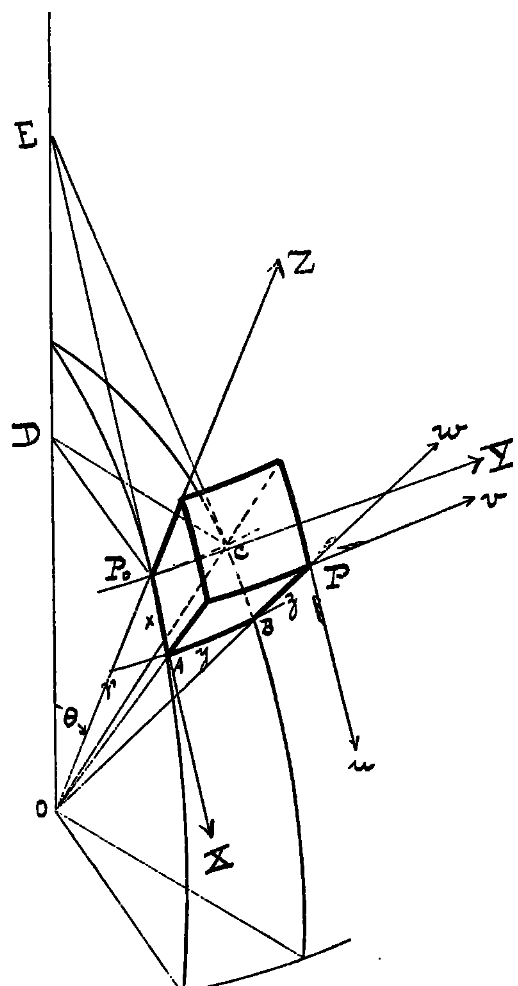
Another set of values may be obtained for the velocities in terms of the primary system X, Y, Z. Let u, v, and w be the velocity-components, and x, y, and z the coordinates, of any one of the cluster of points surrounding the origin P_0, of the two auxiliary systems of Cartesian axes, and let θ, λ, r, be the polar coordinates, with reference to the center and axis of

[1] "Essai sur la theorie des eaux courantes," 1872, part I, §§ 1 and 2, pp., 24–26. Memoires des Savans Etrangers. Tome XXIII. Paris, 1877.

the earth, of this same origin P_0, which now becomes the position of the centroid of the given particle at the time $t = t_0$. It will be supposed that x, y, z are each so small that their squares and products may be neglected. Then resolving u, v, and w parallel to the three axes X, Y, and Z, it will be found that

$$\begin{aligned} \dot{x} &= u + wx/r - vy/r \cot\theta \\ \dot{y} &= v + wy/r + uy/r \cot\theta \\ \dot{z} &= w - vy/r - ux/r \end{aligned} \qquad (1)$$

The verification of these equations is easiest made by comparison with Fig. 2, where the magnitudes of the various angles are indicated.



P = point (x, y, z); $P_0A = x$; $AB = y$; $BP = z$.

$$\left.\begin{aligned} \text{Angle } P_0OA &= \frac{x}{r} \\ \text{Angle } AOB &= \frac{y}{r} \end{aligned}\right\} \qquad \left.\begin{aligned} \text{Angle } P_0DC &= \frac{y}{r\sin\theta} \\ \text{Angle } P_0EA &= \frac{y}{r}\cot\theta \end{aligned}\right\}$$

FIG. 2.

Thus, at P_0, where $x = y = z = 0$, and $x_1 = y_1 = z_1 = 0$,

$$\dot{x}_1 = \dot{x} = u \ ; \quad \dot{y}_1 = \dot{y} = v \ ; \quad \dot{z}_1 = \dot{z} = w,$$

and $\ddot{x}_1, \ddot{y}_1, \ddot{z}_1$ are the time rates of variation of these same quantities with respect to the moving system, or to the polar coordinate system, for the angular velocities imparted to the moving axes were such as to make these remain parallel to the directions in which the polar velocities are measured for a small interval of time near the given instant $t = t_0$. Now, in a small time δt, the centroid of the particle, moves from (θ, λ, r) a distance $w\delta t$ toward the zenith, and changes its angular position by $u/r \cdot \delta t$ southward along the meridian, and by $\frac{v}{r\sin\theta}\,\delta t$ eastward along a parallel. Hence, for example,

$$\delta u = \frac{\partial u}{\partial\theta}\cdot\frac{u}{r}\,\delta t + \frac{\partial u}{\partial\lambda}\cdot\frac{v}{r\sin\theta}\,\delta t + \frac{\partial u}{\partial r}\cdot w\,\delta t + \frac{\partial u}{\partial t}\cdot\delta t$$

and
$$\ddot{x}_1 = \frac{\delta u}{\delta t} = \frac{\partial u}{\partial\theta}\cdot\frac{u}{r} + \frac{\partial u}{\partial\lambda}\cdot\frac{v}{r\sin\theta} + \frac{\partial u}{\partial r}\cdot w + \frac{\partial u}{\partial t}$$

But at the origin,

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial y} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial r} \end{aligned} \qquad (2)$$

and the relative acceleration of the particle southward may be written

$$\ddot{x}_1 = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \equiv \frac{Du}{Dt} \ ,$$

which interpretation will always be attached to the operative symbol D/Dt; that is differentiation along a "stream-line," or along the path of a particle.

In a similar fashion, it may be shown that

$$\ddot{y}_1 = Dv/Dt \quad \text{and} \quad \ddot{z}_1 = Dw/Dt \ ;$$

whence the components of the acceleration of the centroid of a particle are seen to be, respectively,

$$\begin{aligned} \ddot{x} &= \frac{Du}{Dt} - \frac{v^2}{r}\cot\theta + \frac{uw}{r} \\ \ddot{y} &= \frac{Dv}{Dt} + \frac{uv}{r}\cot\theta + \frac{vw}{r} \\ \ddot{z} &= \frac{Dw}{Dt} - \frac{u^2}{r} - \frac{v^2}{r} \ . \end{aligned} \qquad (3)$$

If ϖ_1, ϖ_2, and ϖ_3 be the angular velocities of a particle of the fluid about axes directed southward, eastward, and zenithward, respectively, and, as before, positive when left-handed, the values of these components of the molecular rotation, in terms of the primary Cartesian system XYZ, will be[1]

$$\varpi_1 = \tfrac{1}{2}\left(\frac{\partial\dot{z}}{\partial y} - \frac{\partial\dot{y}}{\partial z}\right); \quad \varpi_2 = \tfrac{1}{2}\left(\frac{\partial\dot{x}}{\partial z} - \frac{\partial\dot{z}}{\partial x}\right); \quad \varpi_3 = \tfrac{1}{2}\left(\frac{\partial\dot{y}}{\partial x} - \frac{\partial\dot{x}}{\partial y}\right).$$

Substituting herein the values of $\dot{x}, \dot{y}, \dot{z}$, from equations (1), and making x, y, and z zero after performing the appropriate partial derivations, it will be found upon combination that

$$\begin{aligned} 2\varpi_1 &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} - \frac{v}{r} \\ 2\varpi_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} + \frac{u}{r} \\ 2\varpi_3 &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{v}{r}\cot\theta \end{aligned}$$

The relative motions of the particles of the fluid are completely determined by the three rates of elongation of lines of particles lying southward, eastward, and zenithward (corresponding to the "stretches" of the theory of elastic solids),

[1] Basset, Treatise on Hydrodynamics, Vol. I, p. 24.

and by the rates of distortion of the faces of a rectangular parallelopiped whose edges at the instant are oriented southward, eastward, and zenithward, respectively. These rates of distortion correspond to the "shears" of the theory of elasticity of solid bodies. The three rates of elongation may be denoted by s_x, s_y, and s_z, and the six rates of distortion, equal in pairs, by σ_{yz}, σ_{zx}, σ_{xy}, and σ_{zy}, σ_{xz}, σ_{yx}. In terms of the primary Cartesian system, these quantities are known to have the values:[1]

$$s_x = \frac{\partial \dot{x}}{\partial x}; \quad \sigma_{yz} = \sigma_{zy} = \frac{\partial \dot{z}}{\partial y} + \frac{\partial \dot{y}}{\partial z};$$

$$s_y = \frac{\partial \dot{y}}{\partial y}; \quad \sigma_{zx} = \sigma_{xz} = \frac{\partial \dot{x}}{\partial z} + \frac{\partial \dot{z}}{\partial x};$$

$$s_z = \frac{\partial \dot{z}}{\partial z}; \quad \sigma_{xy} = \sigma_{yx} = \frac{\partial \dot{y}}{\partial x} + \frac{\partial \dot{x}}{\partial y};$$

or, substituting the expressions for $\dot{x}, \dot{y}, \dot{z}$ from (1), and evaluating at the point (θ, λ, r) as for the molecular rotations, there result the following rates of elongation:

$$s_x = \frac{\partial u}{\partial x} + \frac{w}{r}$$

$$s_y = \frac{\partial v}{\partial y} + \frac{w}{r} + \frac{u}{r}\cot\theta$$

$$s_z = \frac{\partial w}{\partial z}$$

and the following rates of distortion:

$$\sigma_{yz} = \sigma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} - \frac{v}{r}$$

$$\sigma_{zx} = \sigma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{u}{r}$$

$$\sigma_{xy} = \sigma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{v}{r}\cot\theta$$

At the same time the value of the "rate of dilatation" may also be found, or the rate at which a very small portion of the fluid is increasing its volume. This is a time rate, and is expressed by the following:

$$\varepsilon = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = s_x + s_y + s_z$$

whence, upon substitution we have at the point (θ, λ, r),

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{2w}{r} + \frac{u}{r}\cot\theta$$

Kinetical considerations.—Let it be assumed that the stresses are linear functions of the rates of strain, and that the ratio of the tangential traction on any plane to the rate of shear on that plane is the coefficient of friction, μ, already noted. Let the normal tractions be

$$t_{xx}\ , \quad t_{yy}\ , \quad \text{and } t_{zz}$$

on the plane of the prime vertical, the meridian plane, and the horizontal plane of any point, respectively. Let the tangential tractions on the same planes be

$$t_{yz}\ , \ t_{zy}; \quad t_{zx}\ , \ t_{xz}; \quad \text{and } t_{xy}\ , \ t_{yx}\ ;$$

where the first suffix refers to the plane on which the stress is estimated, and the second suffix to the direction of its action. These tractions will be expressed, in terms of the rates of strain, by the following relations:[1]

$$t_{xx} = -p - \tfrac{2}{3}\mu\varepsilon + 2\mu s_x$$
$$t_{yy} = -p - \tfrac{2}{3}\mu\varepsilon + 2\mu s_y$$
$$t_{zz} = -p - \tfrac{2}{3}\mu\varepsilon + 2\mu s_z$$

$$t_{yz} = \mu\sigma_{yz} = t_{zy}$$
$$t_{zx} = \mu\sigma_{zx} = t_{xz}$$
$$t_{xy} = \mu\sigma_{xy} = t_{yx}$$

where p is the "fluid pressure" at any point, or the mean normal pressure over the surface of a particle, and is independent of the intermolecular friction, or of the relative motion of the particles.

The preceding formulas express the intensities of the tractions at the centroid of an elementary parallelopiped, with edges, dx, dy, and dz, respectively. Consequently, the net force in the southward, eastward, and zenithward directions at the centroid, due to the surface tractions, will be found by summing up the components in these directions of the total stresses on all six faces of the parallelopiped. The actual labor of accounting for all these tractions is tedious, owing to the curvatures of the spherical and conical faces of the elementary solid, causing certain of the tractions to have components in certain of the three principal directions at the centroid which would otherwise vanish. For inspection, however, these net forces are here given as follows: The total force on a particle of dimensions dx, dy, dz, due to the surface tractions, when resolved in the southward, eastward, and zenithward directions at the centroid, gives the following components, respectively:

$$\text{Southward} \ldots\ldots \left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial x}(\sin\theta \cdot t_{xx}) + \frac{\partial}{\partial y}(t_{yx}) + \frac{1}{r^2}\frac{\partial}{\partial z}(r^2\, t_{zx}) \right\} dx\,dy\,dz - (t_{yy}\cot\theta + t_{xz})\frac{dx\,dy\,dz}{r}$$

$$\text{Eastward} \ldots\ldots \left\{ \frac{\partial}{\partial y}(t_{yy}) + \frac{1}{r^2}\frac{\partial}{\partial z}(r^2\, t_{zy}) + \frac{1}{\sin\theta}\frac{\partial}{\partial x}(\sin\theta \cdot t_{xy}) \right\} dx\,dy\,dz - t_{yz}\frac{dx\,dy\,dz}{r}$$

$$\text{Zenithward} \ldots\ldots \left\{ \frac{1}{r^2}\frac{\partial}{\partial z}(r^2 t_{zz}) + \frac{1}{\sin\theta}\frac{\partial}{\partial x}(\sin\theta \cdot t_{xz}) + \frac{\partial}{\partial y}(t_{yz}) \right\} dx\,dy\,dz - (t_{xx} + t_{yy})\frac{dx\,dy\,dz}{r}$$

It was before remarked that the equations of motion desired are to be found by equating the "expressed force" of a particle to the sum of the "impressed forces," and of the above given total force due to the surface tractions. If ρ is the density of the fluid, $\rho\, dx\, dy\, dz$ will be the mass of the particle, which multiplied by the acceleration (3) gives the "expressed force" in any direction. Hence, remembering the convention that the only impressed forces are those that like gravitation can be expressed in terms of a force-function that is dependent upon the radius only, and after reduction according to the previously deduced formulas and division by the mass of the particle, there result the following equations of motion:

[1] Lamb, Hydrodynamics, § 31 (where, however, $\sigma_{\lambda r} = 2f = \sigma_{r\lambda}$, etc.).

[1] Lamb, Hydrodynamics, § 284.

$$\frac{Du}{Dt}-\frac{v^2}{r}\cot\theta+\frac{uw}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\varepsilon}{\partial x}+\Delta^2 u+\frac{2}{r}\frac{\partial w}{\partial x}-\frac{2\cot\theta}{r}\frac{\partial v}{\partial y}-\frac{u}{r^2\sin^2\theta}\right]$$

$$\frac{Dv}{Dt}+\frac{uv}{r}\cot\theta+\frac{vw}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\varepsilon}{\partial y}+\Delta^2 v+\frac{2}{r}\frac{\partial w}{\partial y^1}+\frac{2\cot\theta}{r}\frac{\partial u}{\partial y}-\frac{v}{r^2\sin^2\theta}\right] \qquad (4)[1]$$

$$\frac{Dw}{Dt}-\frac{u^2+v^2}{r}=\frac{\partial F}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\varepsilon}{\partial z}+\Delta^2 w-\frac{2}{r}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-2w+\frac{2u\cot\theta}{r^2}\right]$$

where the Laplacian operator Δ^2, as applied to the modified polar coordinates, has the following interpretation:

$$\Delta^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\cot\theta}{r}\frac{\partial}{\partial x}+\frac{2}{r}\frac{\partial}{\partial z}\,.$$

To the above three general equations of motion must be joined the so-called equation of continuity,

$$\frac{D\rho}{Dt}+\rho\varepsilon=0,$$

found, as usual, by equating the rate of increase of the mass in an elementary space due to the total rate of flow through the boundary to the rate of increase of the mass in that same space due to increase in density of the fluid.

It is also necessary to specify the relation between pressure and density at any point. Of the many hypotheses which might be formed as to this relation, the two simplest are: (*a*) that of homogeneity, and (*b*) that of uniform distribution of temperature. The hypothesis (*a*) means that the density ρ shall be considered constant, or that the rate of dilation shall be zero, thereby reducing the equation of continuity to

$$\varepsilon=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{u}{r}\cot\theta+\frac{2w}{r}=0$$

In the case of the earth the total height of the atmosphere if it were homogeneous would be

$$p_0/g\rho_0=5 \text{ miles, nearly,}$$

where p_0 and ρ_0 are the normal pressure and density at sea level; and the rate of decrement of the pressure with increase in altitude would be constant.

The hypothesis (*b*) is equivalent, analytically, to the isothermal law

$$\frac{p}{p_0}=\frac{\rho}{\rho_0}$$

These two hypotheses are manifestly at variance with the known properties of the atmosphere, whose density diminishes rapidly on ascent, and whose vertical temperature decrement averages 1° F. per 300 feet of altitude,[2] and they must be discarded, unless for special purposes, or unless only a rough degree of approximation is desired.

The hypothesis (*c*) that the equilibrium of the atmosphere is not static but convective, and is kept up by ascending and descending columns of air which fall and rise in temperature through their expansion and compression without sensible change of the total quantity of heat, gives better promise. This is equivalent to the so-called adiabatic law, or the law governing the expansion of a gas without transmission of heat,

$$\frac{p}{p_0}=\left(\frac{\rho}{\rho_0}\right)^\gamma$$

where $\gamma=1.406=$ ratio of specific heat of air at constant pressure to specific heat at constant volume.[3]

The total height of an atmosphere following the adiabatic law would be about 17 miles; the pressure would be reduced to one half its sea-level value at an altitude of slightly over 3 miles; to 1/10 at 8.4 miles, to 1/100 at about 13 miles, and to 1/5000 at an altitude of about 16 miles, if gravity be considered constant, the atmosphere in a stable condition, and the effect of the earth's rotation neglected.

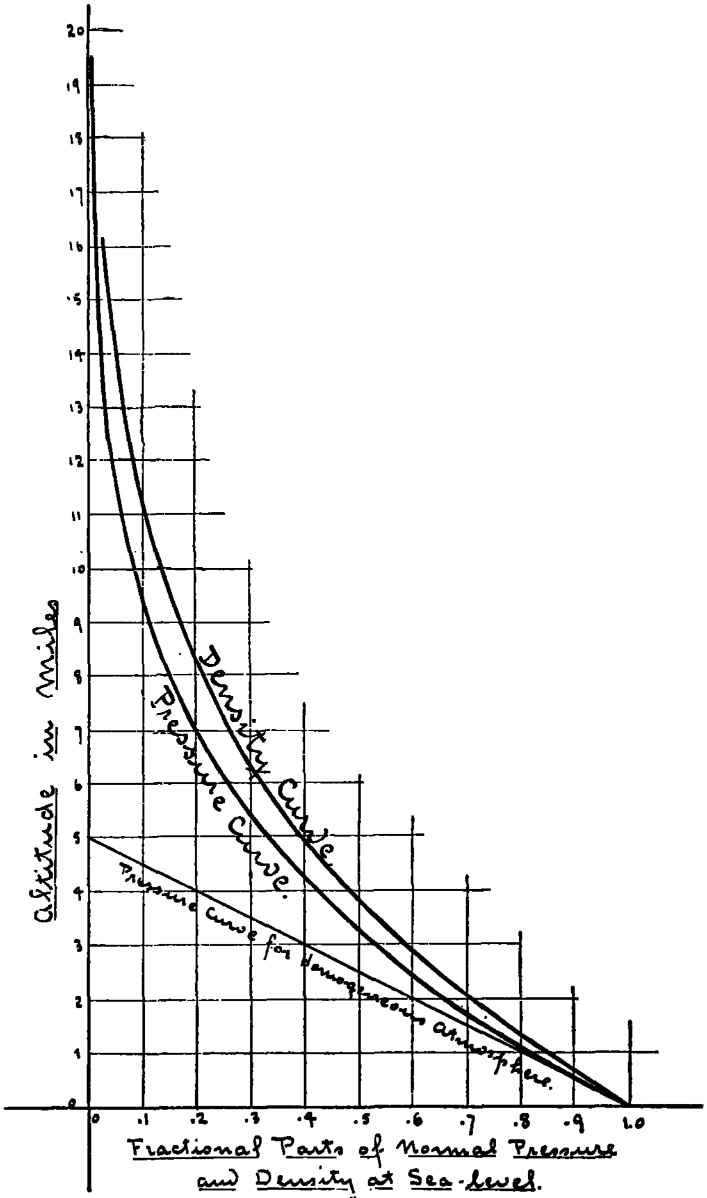


FIG. 3.—Pressure graph of atmosphere, on hypothesis of uniform decrease of temperature at rate of 1° F. per 100 yards ascent. $\nu=1.2$.

Even this hypothesis does not represent the observed state of the atmosphere accurately, since it means a rate of decrease of 1.6° F. per 100 yards increase in altitude, instead of

[1] If these equations are preferred in terms of derivatives with respect to terrestrial polar coordinates (θ, λ, r,) instead of local coordinates (x, y, z), the change may be readily effected by the aid of equations (2).

[2] Davis, Elementary Meteorology, § 36.

[3] Wood, Thermodynamics, p. 54.

the 1° F. which the presence of moisture in the air and other causes conspire to determine.

As the observed rate of decrease of temperature lies between hypotheses b and c, let it be assumed (d) that

$$\frac{p}{p_0}=\left(\frac{\rho}{\rho_0}\right)^{\nu}. \tag{5}$$

It is easy to so determine ν that the temperature shall decrease at the observed average rate of 1° F. per 100 yards. Thus considering g as constant, and the atmosphere in a state of stable equilibrium,

$$p=\int_z -\rho\, g\, dz, \text{ whence } dp=-\rho\, g\, dz, \tag{6}$$

where z is the altitude above the surface of the earth. Eliminating ρ from (5) and (6) and integrating, there results

$$p_0^{\frac{\nu-1}{\nu}}-p^{\frac{\nu-1}{\nu}}=\frac{\nu-1}{\nu}\,\frac{g\,\rho_0}{p_0^{1/\nu}}\,z,$$

where $p=p_0$ when $z=0$. But if τ is the absolute temperature of the atmosphere (temperature in Fahrenheit degrees + 461°, or in Centigrade degrees + 274°) and if the air is a perfect gas it must satisfy the well-known relation

$$\frac{p}{\rho\,\tau}=\frac{p_0}{\rho_0\,\tau_0}=1{,}713$$

in British units, where p = poundals/ft.^{2} and ρ = lbs./ft.^{3}; or

$$\frac{p_0}{\rho_0\,\tau_0}=2{,}864{,}000$$

in C. G. S. units, when p = dynes/cm.^{2} and ρ = gr./cm.^{3}; τ being measured in Fahrenheit degrees in the former case, in Centigrade degrees in the latter.

Making use of equations (5), (6), and (7), ν may be determined so as to make

$$\frac{d\tau}{dz}=-\frac{1}{300}\text{ F.°/ft.},$$

so as to accord with the observed value of the rate of diminution of temperature. The value thus found is approximately $\nu=1.2$.

The total height of an atmosphere satisfying this law,

$$\frac{p}{p_0}=\left(\frac{\rho}{\rho_0}\right)^{1.2},$$

would be about 30 miles; the pressure would be reduced to one-half its value at sea level at an altitude of $3\frac{1}{4}$ miles, to 1/10 at $9\frac{1}{2}$ miles, to 1/100 at 16 miles, and to 1/5000 part at an altitude of about 23 miles. These values are somewhat affected by the diminution of the value of g with increasing altitude and by the presence and increase of the centrifugal force due to the rotation of the earth.

The agreement with the known properties of the atmosphere is much closer than under the preceding hypotheses; it is sensibly exact at lower altitudes, and although the atmosphere is believed to extend in a state of extreme tenuity to a height considerably greater than 100 miles, no significant pressure would be encountered at an altitude of 45 or 50 miles. It is estimated that at an altitude of 30 miles the pressure is but $\frac{1}{200}$ of an inch of mercury[1] (about 1/6000 of the pressure at sea level), a discrepancy of less than 20 per cent in altitude between the calculated and estimated values at this low pressure.

A graph of the pressure and density distributions under this law is given in Fig. 3.

If the law of distribution of the absolute temperature be assumed (e) to be any other function of the radius or altitude, and if need be also of the latitude, the relation between pressure and density may be found by eliminating the temperature between this assumed law and equation (7).

In the equations as thus far discussed, no specific reference has been made to the rate of rotation of the earth, which may be regarded as possessed of a constant angular velocity

$$\omega=\frac{2\pi}{86{,}164}=0.0000729\text{ Sec.}^{-1}$$

about its polar diameter. An exactly sufficient allowance for this rotation will be made by substituting for (v) the value ($v+\omega r\sin\theta$) wherever v appears in the left-hand members of the three equations of motion, except in the interpretation of the operator D/Dt, which refers to an operation relative to the earth.

Thus, the corrected equations of motion (4) become

$$\begin{aligned}\frac{Du}{Dt}-\frac{v^2}{r}\cot\theta+\frac{uw}{r}-2v\,\omega\cos\theta-\omega^2 r\sin\theta\cos\theta&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\Big[\cdots\cdots\Big]\\ \frac{Dv}{Dt}+\frac{uv}{r}\cot\theta+\frac{vw}{r}+2u\,\omega\cos\theta+2w\,\omega\sin\theta&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\Big[\cdots\cdots\Big]\\ \frac{Dw}{Dt}-\frac{u^2+v^2}{r}-2v\,\omega\sin\theta-\omega^2 r\sin^2\theta&=\frac{\partial F}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\Big[\cdots\cdots\Big]\end{aligned} \tag{8}$$

Now if the atmosphere is in a state of relative rest, so that $u=v=w=0$, these equations reduce to

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x}&=\omega^2 r\sin\theta\cos\theta\\ \frac{1}{\rho}\frac{\partial p}{\partial y}&=0\\ \frac{1}{\rho}\frac{\partial p}{\partial z}&=\frac{\partial F}{\partial z}+\omega^2 r\sin^2\theta\end{aligned}$$

the resultant of which forces must be normal to the earth's spheroid, or to the level surface of the earth. But, according to the assumption whereby the eccentricity of the earth was neglected and the earth considered spherical, that level surface must be spherical, and

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x}&=\quad 0\\ \frac{1}{\rho}\frac{\partial p}{\partial y}&=\quad 0\\ \frac{1}{\rho}\frac{\partial p}{\partial z}&=-g\end{aligned}$$

Hence, to give consistent results with the arbitrarily chosen spherical equipotential surface, the quantity ($\omega^2 r\sin\theta\cos\theta$)

[1] Davis, op. cit., p. 13.

of the first equation must be discarded, and for $\left(\frac{\partial F}{\partial z}+\omega^2\sin^2\theta\right)$ of the third equation must be substituted $(-g)$.[1]

Still g will vary with the altitude; and if g_0 is its value at the radius a, then

$$g = g_0 a^2/r^2$$

for points exterior to the earth.

Now the mean mass-altitude of the atmosphere, or the altitude of the centroid of a vertical column of air, according to the suggested relation between pressure and density, is about $4\frac{1}{4}$ miles. Therefore, if the mean atmospheric radius be considered as $a = 3{,}959 + 4 = 3{,}963$ miles, or 6,377 kilometers, an error of 1 per cent in terms containing $1/r^2$, and of only one-half of 1 per cent in terms containing $1/r$, will be made by considering r constant and equal to a at as great a distance from the surface of the earth as 24 miles, at which altitude the density of the air is inappreciable and its effect on the general circulation a minimum; whereas at the surface of the earth, where the density is a maximum, the error induced by considering r constant will not sensibly exceed one-fifth of 1 per cent in terms containing $1/r^2$, nor one-tenth of 1 per cent for terms in $1/r$. The quantity g is the term most affected by this change; yet the error is small and constant for any stratum concentric with the earth. It will therefore not give rise to any disturbing pressure gradient, and it may be concluded that all necessary accuracy will be obtained, if we consider $r = a = 3{,}963$ miles, wherever that quantity enters explicitly in the equations of motion.[1]

Certain terms of these equations are always negligibly small. Thus the actual component of the wind velocity parallel to the surface of the earth can not well reach as high a value as 200 miles per hour; yet if that were possible the quantity $(v^2+u^2)/a$ of the third of equations (8) would not exceed 0.000008 ft./sec.^{2}, and the maximum value of the quantity $2\,\omega\, v\sin\theta$ in the same equation would be less than 0.002 ft./sec.^{2}, both of which may be neglected in comparison with the value of g. In that same equation, the quantity $2\frac{\mu}{\rho}\frac{w}{a^2}$ may be similarly neglected. Several other terms may be negligibly small if the territory covered by the particles under analysis does not include the terrestrial poles; but the resulting changes are obvious.

Making only those alterations specifically noted, the general equations of motion are obtained in their final forms as follows:

$$\frac{Du}{Dt}-\frac{v^2}{a}\cot\theta+\frac{uw}{a}-2\,v\,\omega\cos\theta+\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\varepsilon}{\partial x}+\Delta^2 u+\frac{2}{a}\frac{\partial w}{\partial x}-\frac{2\cot\theta}{a}\frac{\partial v}{\partial y}-\frac{u}{a^2\sin^2\theta}\right]$$

$$\frac{Dv}{Dt}+\frac{uv}{a}\cot\theta+\frac{vw}{a}+2\,u\,\omega\cos\theta+2\,w\,\omega\sin\theta+\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\varepsilon}{\partial y}+\Delta^2 v+\frac{2}{a}\frac{\partial w}{\partial y}+\frac{2\cot\theta}{a}\frac{\partial u}{\partial y}-\frac{v}{a^2\sin^2\theta}\right]$$

$$\frac{Dw}{Dt}+g+\frac{1}{\rho}\frac{\partial p}{\partial z}=\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\varepsilon}{\partial z}+\Delta^2 w-\frac{2}{a}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{2\,u\cot\theta}{a^2}\right],$$

where the operator D/Dt and the Laplacian operator Δ^2, and the dilatation ε, have already been defined. To these equations are to be added the equation of continuity of the fluid, and the relation between pressure and density. In the case of frictionless or "perfect" fluids, the second members will be each made zero.

NOTE.—In the United States Coast Survey Report for 1875, p. 372, Ferrel gives in his equations (8) formulas differing from equations (8) of this present paper by the factor 2. Thus, Ferrel's equations would give the following instead of our equation (8):

$$\frac{Du}{Dt}-\frac{v^2\cot\theta}{r}+\frac{{}^*2\,uw}{r}-2\,\omega\,v\cos\theta-\omega^2 r\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\dots$$

$$\frac{Dv}{Dt}+\frac{{}^*2\,vw}{r}+\frac{{}^*2\,uv\,\cot\theta}{r}+2\,\omega\,w\sin\theta+2\,\omega\,u\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\dots$$

$$\frac{Dw}{Dt}-\frac{u^2+v^2}{r}-\omega^2 r\sin^2\theta-2\,\omega\,v\sin\theta=\frac{\partial F}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\dots$$

The superfluous factors 2* are errors in Ferrel's equations, introduced in the passage from equations (7) of his paper to (8).

In fact, the accelerational terms of his equation (7) are

$$D_t^2\theta\ ,\quad D_t^2\lambda\ ,\quad \text{and}\ D_t^2 r\ ,$$

or which is equivalent,

$$D_t(u/r)\ ,\quad D_t(v/r\sin\theta)\ ,\quad \text{and}\ D_t w\,.$$

This symbol D is, in the paper referred to, treated as the symbol of partial derivation with respect to the time, a similar notation being used for partial derivatives with respect to the space coordinates. In this lies Ferrel's error, for D_t is identical with the operator D/Dt of this paper, and is a total derivative, defined in terms of the quantities appearing in the equations by

$$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$$

By introducing this correct interpretation of the operator D_t the equations deduced from (7) of Ferrel's paper will be exactly equivalent to our equations (8).

This error has already been pointed out by Sprung,[2] who derived the equations in a different manner.[3]

[1] Compare Ferrel, "Recent Advances in Meteorology," Report of the Chief Signal Officer, U. S. War Department, 1885, Part II, p. 184.

[1] At that altitude $g_0 = 32.08$ ft./sec.^{2} Cf. Thomson and Tait, "Treatise on Natural Philosophy," Vol. I, Part 2, § 801.

[2] Meteorologie, 1885, p. 208.

[3] It had also been recognized by Ferrel (see his Meteorological Researches No. III, 1882, p. 46, or Report of Coast and Geodetic Survey for 1881, p. 268) and does not reappear in his Recent Advances, Washington, 1885, but has unfortunately been copied, inadvertently, by several subsequent authors.—ED.